

A Simple Inductive Approach to the Problem of Convergence of Cluster Expansions of Polymer Models

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We explain a simple inductive method for the analysis of the convergence of cluster expansions (Taylor expansions, Mayer expansions) for the partition functions of polymer models. We give a very simple proof of the Dobrushin–Kotecký–Preiss criterion and formulate a generalization usable for situations where a successive expansion of the partition function has to be used.

KEY WORDS: Cluster expansion; Taylor expansion; Mayer expansion; polymer models.

In this short note we explain a new and simple inductive method for the analysis of the convergence of cluster expansions of so called *polymer models*. The notion of a polymer model goes back to Gruber and Kunz⁽¹⁾ (see also refs. 2–10 for some fundamental contributions to the theory of cluster expansions; it is impossible to give an exhaustive account here. A good review is given in ref. 4) and it was Dobrushin⁽¹¹⁾ who first fully exploited the fact (already pointed out by Gruber and Kunz) that cluster expansions of these models are actually Taylor expansions of the logarithm of the partition function, taken with respect to the fugacities of the considered polymers. Our new approach was already used in a recent paper⁽¹²⁾ (and a very similar approach was more recently used in refs. 13 and 14) but here we further simplify and streamline the argument and extend it in order to be applicable also to partially expanded polymer models and thus multi-scale expansions. This is important, e.g., in the study of models with random

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impurities (see, e.g., refs. 15 and 16) and in other situations where the “expandability” of a given “large” polymer (contour) Γ may be clarified only after expanding all the contours “smaller than Γ .” In these situations, sequential expansions are indispensable and it is thus important to know that even in the case of an ordinary polymer models, the sequential approach gives an equally good control of the situation as the expansion “at once.”

1. POLYMER MODELS. THE DOBRUSHIN–KOTECKÝ–PREISS CRITERION

Let \mathcal{P} be a set whose elements $P_1, \dots, P_{|\mathcal{P}|}$ are called *polymers* (we should emphasize that the name *polymer* is used solely for historical reasons and may be misleading. For our present purposes, the P_i are just labels for the elements of the finite set and we might as well label them in the standard way by integers). We suppose given a binary symmetric relation c of “compatibility” between the polymers. This means that in the product $\mathcal{P} \times \mathcal{P}$ we give a symmetric subset of “compatible” pairs (note that this compatibility relation encodes all what remains of the structural properties of the physical models). Two polymers which are not compatible are said to be incompatible and we write $P_1 c P_2$, $P_1 \iota P_2$ when $\{P_1, P_2\}$ is a compatible resp. an incompatible pair. We will assume that $P \iota P$ for all $P \in \mathcal{P}$.

Following Dobrushin,⁽¹¹⁾ we will associate to each polymer P a complex variable w_P and introduce the *polymer partition function*

$$Z \equiv Z_{\mathcal{P}} \equiv Z_{\mathcal{P}, w} = \sum_{\{P_1, \dots, P_n\}_c: P_i \in \mathcal{P}} \prod_{i=1}^n w_{P_i} \quad (1)$$

where the sum is over all families $\{P_1, \dots, P_n\}_c$ of pairwise compatible polymers from \mathcal{P} . The $n=0$ term in (1) (no polymers at all) is set equal to 1. Note that $Z_{\mathcal{P}}$ is a function of the $|\mathcal{P}|$ complex variables $w_1, \dots, w_{|\mathcal{P}|}$.

Remark. In many applications, there is a spatial structure such that it is possible to associate a “support” $\text{supp } P$, namely a finite subset of \mathbb{Z}^d , to a polymer P . In those cases the compatibility $P c \tilde{P}$ is usually just some geometrical property of the supports, typically polymers are compatible if their supports do not intersect. Also the polymer activities arise as some simple functions depending on the “shape” of the polymer, the temperature and interaction potentials. To avoid confusion, let us stress, however, that in the abstract polymer models we study now, we do not consider these “physical” activities, but all polymers activities w_P are now treated as independent complex variables. The relation to the physical activities is made

only later when thermodynamic functions of the abstract polymer model are evaluated at the physical values of the activities.

As usual we are interested in the computation of the logarithm of the partition function. We can write its Taylor series around zero

$$\log Z = \log Z_{\mathcal{P}, w} = \sum_{I \in \mathcal{I}(\mathcal{P})} w_I \quad \text{with} \quad w_I = C_I \prod_{P \in \text{supp } I} w_P^{I_P} \quad (2)$$

where the sum is over the collection $\mathcal{I}(\mathcal{P})$ of all "multi-indices" I (integer valued functions on \mathcal{P}). The Taylor coefficients C_I , $I = (I_P)_{P \in \mathcal{P}}$ are

$$C_I = (I_{P_1}! \cdots I_{P_N}!)^{-1} \left. \frac{\partial^{I_{P_1} + \cdots + I_{P_N}} \log Z_{\mathcal{P}}}{\partial^{I_{P_1}} w_{P_1} \cdots \partial^{I_{P_N}} w_{P_N}} \right|_{w_{P_i} \equiv 0} \quad (3)$$

where the derivative of $\log Z_{\mathcal{P}}$ is taken at $\{w_{P_i} \equiv 0, P_i \in \mathcal{P}\}$, $\mathcal{P} = \{P_1, \dots, P_N\}$. This purely analytic way of defining the coefficient C_I was introduced by Dobrushin.⁽¹¹⁾ Traditionally, these coefficients were defined in combinatorial terms as sums over connected graphs with vertices labelled by the polymers that results from a formal expansion of the logarithm in power series. Convergence proofs were then based on ingenious combinatorial methods which contributed to the (bad?) reputation of the cluster expansion method as a very complicated tool (for an excellent exposition and many of the earlier references, see ref. 4). We hope to show here to what extent this reputation is unjustified.

A multi-index I on \mathcal{P} can be regarded as a collection of polymers where multiple copies of a single polymer P are allowed. Then the non-negative integer I_P represents the multiplicity of P in I . Write $I \equiv (\mathcal{S}, I_{\mathcal{S}})$ where $\mathcal{S} = \text{supp } I$ is the "support of I ," $\mathcal{S} = \{P \in \mathcal{P} : I_P \geq 1\}$. Given any subset $\mathcal{S} = \{P_1, \dots, P_n\} \subset \mathcal{P}$ and activities w_P , we denote by $w^{\mathcal{S}}$ modified activities such that $w_P^{\mathcal{S}} = w_P$, $P \in \mathcal{S}$ and $w_P^{\mathcal{S}} = 0$, $P \notin \mathcal{S}$.

Notice that we have for any function F given by a power series in variables $\{w_P, P \in \mathcal{P}\}$ the relation $\partial_{\mathcal{S}} F\{w_P, P \in \mathcal{P}\} = \partial_{\mathcal{S}} F\{w_P, P \in \mathcal{S}\}$ where the symbol $\partial_{\mathcal{S}}$ denotes a derivative (of any order) taken w.r.t. the variables $\{w_P, P \in \mathcal{S}\}$ at $\{w_P \equiv 0, P \in \mathcal{P}\}$. Hence the coefficients C_I are functions of the multi-index I (on the system of polymers $\{P \in \text{supp } I\}$, with the compatibility relation $P c P'$) only.

In fact, nonzero values of C_I appear only for *indecomposable* multi-indices I ; decomposability means that there exists a partition of $S = \text{supp } I$ into two sets $S = S_1 \cup S_2$ such that $\{P_1, P_2\}$ is a compatible pair $\forall P_1 \in S_1$,

$P_2 \in S_2$. We will also use a name *cluster* for such an indecomposable multi-index (collection of polymers) I . The collection of all clusters on \mathcal{P} will be denoted by $\mathcal{C} = \mathcal{C}(\mathcal{P})$.

Indeed, analyze the coefficient C_I for a decomposable index $I = (S, I_S)$. If $S = S_1 \cup S_2$ with $S_1 \neq \emptyset$, $S_2 \neq \emptyset$, $S_1 \cap S_2 = \emptyset$ and every pair (P_1, P_2) with $P_1 \in S_1$, $P_2 \in S_2$ is compatible, then $\log Z_{S, w} = \log Z_{S_1, w} + \log Z_{S_2, w}$ and so $C_I = 0$, by (3).

The following theorem appeared first in ref. 12, and we will give a even a bit simplified proof of it that is based on a very simple induction argument. Put

$$L = L(\delta) = \sup_{x \in (0, \delta)} \left\{ \frac{-\log(1-x)}{x} \right\} = \frac{-\log(1-\delta)}{\delta} \quad (4)$$

In usual applications, δ will be small and so $L = 1 + O(\delta)$.

Theorem 1. Assume that there is a function $a_P \geq 0$ on \mathcal{P} such that

$$|w_P| e^{a(P)} \leq \delta \quad (5)$$

holds for any polymer $P \in \mathcal{P}$. Moreover, assume that for any polymer $Q \in \mathcal{P}$

$$\sum_{P \uparrow Q} |w_P| e^{a(P)} \leq \frac{a(Q)}{L} \quad (6)$$

where L is from (4). Then, for any polymer $Q \in \mathcal{P}$, the following bound holds for the sum over all clusters (connected multi-indices) $I \in \mathcal{C}(\mathcal{P})$ containing Q :

$$\sum_{I \in \mathcal{C}(\mathcal{P}): I \ni Q} |w_I| \leq L |w_Q| e^{a(Q)} \quad \text{with} \quad w_I = C_I \prod_{P \in \text{supp } I} w_P^{I_P} \quad (7)$$

As a consequence one also has a bound, for the sum over all clusters I that are incompatible with a selected polymer $Q \in \mathcal{P}$

$$\sum_{I \in \mathcal{C}(\mathcal{P}): I \not\uparrow Q} |w_I| \leq a(Q) \quad (8)$$

Remark. The criterion in the theorem is similar in spirit, but neither weaker nor stronger than that of Kotecký and Preiss⁽¹⁷⁾ (see also ref. 18 for

a good exposition). It is, however, possible to replace condition (6) by the condition

$$\sum_{P \ni Q} -\log(1 - |w_P| e^{a(P)}) \leq a(Q) \tag{6'}$$

and the consequence (7) by

$$\sum_{I \in \mathcal{C}(\mathcal{P}): I \ni Q} |w_I| \leq -\log(1 - |w_Q| e^{a(Q)}) \quad \text{with } w_I = C_I \prod_{P \in \text{supp } I} w_P^{I_P} \tag{7}$$

(condition (5) can then be dropped). The statement of the theorem is then equivalent to Dobrushin’s condition from ref. 11 and *weaker* than the Kotecký–Preiss condition. The inductive proof below carries over without change. In practice, the original formulation (6), (7) is probably easier to use. We owe this observation to a discussion with Alan Sokal on the relation of this work to his recent paper,⁽¹³⁾ where the equivalent statement is also proven.

Remarks. (a) If polymers P are just points of the lattice \mathbb{Z}^v and $\#\{j : j \text{ } i\} \leq k$ for each i then the condition $|w_i| \leq e^{-a}a/k$, i.e., $\sum_{j \text{ } i} |w_j| \leq ae^{-a}$ implies $\sum_{I \ni i} |w_I| \leq a$.

(b) If polymers P are “bonds” $b = \{i, j\}$ and compatibility of bonds means just their non-intersection, then the condition (6) (valid, e.g., if $\sum_{b \ni i} |w_b| \leq e^{-a}a/2$) implies $\sum_{I \ni b} |w_I| \leq a$.

(c) For the low temperature Peierls contours^(19–22) of the two dimensional Ising model, a natural choice of the function a is $a(P) = a|P|$ with $a = 2\beta J - C$ and a suitable C . Then the condition (6)

$$\sum_{P \neq \emptyset} e^{-2\beta J|P|} e^{a|P|} \leq 4 \sum_{n \geq 4} 3^n e^{-(2\beta J - a)n} \leq a$$

is obviously verified for $2J\beta$ sufficiently larger than $\log 3$ (with suitable C).

(d) An important application of the convergence result occurs when the polymer model results from a contour representation in the context of the Pirogov–Sinai theory.^(23–25) In that case the polymers represent local deviations from some ground state configuration g . To each polymer P one then associates a subset of the lattice, $\underline{P} \subset \mathbb{Z}^d$. The incompatibility relation usually refers then to a sufficient distance between the corresponding “supports.” The function $a(P)$ should then be chosen proportional to the volume of the support of P chosen as $a(P) = a|\underline{P}|$. In such a situation the convergence of the polymer expansion guarantees the existence of a Gibbs

state μ_g corresponding to the ground state g in the following sense: Let $S_g(R)$ be the set of spin configurations such that there exists a point x at a distance less than R from the origin such that one can reach infinity from x on a path along which the configuration g is realized. Let $S_g = \bigcup_{R < \infty} S_g(R)$. Then $\mu_g(S_g) = 1$. To see this, note that $S_g(R)$ is not realized only if there is a contour P whose interior contains the ball of radius R . We call \mathcal{P}_R the collection of all such polymers. Then by standard arguments we have that

$$1 - \mu_g(S_g(R)) \leq \sum_{P \in \mathcal{P}(R)} w_P e^{\sum_{I^* \cap P} w_{I^*}} \leq \sum_{P \in \mathcal{P}(R)} w_P e^{d(P)}$$

which will go to zero if R goes to infinity. If one has several such polymer models this means that a rigorous control of the phase coexistence is established.

(e) In many situations, in particular when polymers are geometric objects, there is a natural notion of the *size* of the polymer, like, e.g., their *diameter*. In such a situation a stronger variant of (6) gives an exponential decay of the correlations in the given polymer model. Namely, the statement (8) gives also an information on the *decay* of the terms $w_I = C_I \prod_{P \in \text{supp } I} w_P^I$. Take $\tilde{w}_P = w_P e^{d(P)}$ with $d(P) = C \text{diam } P$ and require (6) to hold *even for* \tilde{w}_P . Then (8) reads

$$a(Q) \geq \sum_{I \subset Q} |\tilde{w}_I| \geq \sum_{I \subset Q} |w_I| e^{\sum_{P \in \text{supp } I} d(P)} \geq \sum_{I \subset Q} |w_I| e^{C \text{diam supp } I}$$

This tells us that the sum $\sum_I^N |w_I|$ taken over clusters of diameter at least N is of order at most e^{-dN} , and only those terms w_I appear in the formulas expressing the correlation between two cylinder events having distance $\geq N$. In the case of two dimensional low temperature Ising contours a convenient choice of a, d is such that $a + d$ is suitably smaller than $2\beta J - \log 3$. Then the correlation length is proven to be of the order $1/d$ (or less).

Proof of Theorem 1. Our proof uses, following ref. 12, an induction over the cardinality $|\mathcal{P}|$ of the system \mathcal{P} of all available polymers. Suppose that we already have (by induction assumption) the bound (8), with $w_I = C_I \prod_{P \in \text{supp } I} w_P^I$, for the sum of Taylor coefficients of any polymer model employing a *smaller* number of polymers than $|\mathcal{P}|$. (It trivially holds for a model employing no polymers at all.) Then we want to prove the same bound for a model constructed over \mathcal{P} .

Select a polymer Q , denote by $\mathcal{P} \setminus Q = \{P \in \mathcal{P} : P \subset Q\}$ and consider the partition functions $Z_{\mathcal{P} \setminus Q}$ and $Z_{\mathcal{P} \setminus\setminus Q}$ of the model “without Q ” resp. “employing only polymers compatible with Q ”

$$Z_{\mathcal{P} \setminus Q} = \sum_{\{P_1, \dots, P_m\}_c, P_i \in \mathcal{P}, Q \neq \{P_i\}} \prod_i w_{P_i}; \quad Z_{\mathcal{P} \setminus\setminus Q} = \sum_{\{Q, P_1, \dots, P_m\}_c, P_i \in \mathcal{P}} \prod_i w_{P_i}$$

We can obviously decompose the partition function for the set \mathcal{P} in the form

$$Z_{\mathcal{P}} = Z_{\mathcal{P} \setminus Q} + w_Q Z_{\mathcal{P} \setminus\setminus Q}$$

by writing first the sum over all terms that do not contain Q and then placing Q and summing over all remaining collections compatible with Q . Taking the logarithm we get

$$\log Z_{\mathcal{P}} = \log Z_{\mathcal{P} \setminus Q} + \log \left(1 + w_Q \frac{Z_{\mathcal{P} \setminus\setminus Q}}{Z_{\mathcal{P} \setminus Q}} \right) \tag{9}$$

Since the first summand here counts the sum over all clusters that do not make use of Q , the second term is necessarily equal to the sum of all clusters containing Q , i.e., the sum we want to control in (7).

On the other hand, the term $Z_{\mathcal{P} \setminus\setminus Q} / Z_{\mathcal{P} \setminus Q}$ appearing in the second logarithm is already “under control” because it uses partition functions of polymer models with less than $|\mathcal{P}|$ polymers. That is to say we have on the one hand that

$$\log \left(1 + w_Q \frac{Z_{\mathcal{P} \setminus\setminus Q}}{Z_{\mathcal{P} \setminus Q}} \right) = \sum_{I \in \mathcal{C}(\mathcal{P}): I \ni Q} w_I \tag{10}$$

and on the other hand

$$\log \left(1 + w_Q \frac{Z_{\mathcal{P} \setminus\setminus Q}}{Z_{\mathcal{P} \setminus Q}} \right) = \log \left(1 + w_Q \exp \left(- \sum_{I^*} w_{I^*} \right) \right) \tag{11}$$

where the sum $\sum_{I^*} w_{I^*}$ is precisely over all the clusters I^* , from the $\mathcal{P} \setminus Q$ model, which are *incompatible* with Q , i.e., which contain some polymer \tilde{Q} incompatible with Q . The sum $\sum_{I^*} w_{I^*}$ can be estimated, using the induction assumption (7), for any \tilde{Q} (notice that the clusters I^* are taken from a “smaller,” $\mathcal{P} \setminus Q$ model) as

$$\sum_{I^*} |w_{I^*}| \leq L \sum_{\tilde{Q} \ni Q} |w_{\tilde{Q}}| e^{a(\tilde{Q})} \leq a(Q) \tag{12}$$

because of (6). Now we will use the following important fact: Consider the Taylor expansion in the variables w_Q, w_{I^*} of the function $\log(1 + w_Q \exp(-\sum_{I^*} w_{I^*}))$ and replace all the coefficients in the resulting sum (of products of w_Q and w_{I^*}) by their absolute values. Use the following simple observation.

Lemma. Denote by $f < g$ the relation, between functions of variables x_1, x_2, \dots, x_n , defined by the requirement that absolute values of all Taylor coefficients of f at $x_i \equiv 0$ are bounded from above by the corresponding positive Taylor coefficients of g . For any monomial $y_j = a_j \prod x_i^{N_i^j}$ denote by $\tilde{y}_j = |a_j| \prod x_i^{N_i^j}$. Then, for any choice $\{y_j\}$ of monomials we have the relation, interpreting both sides as functions of $\{x_i\}$,

$$\log\left(1 + x_1 \exp\left(\sum_j y_j\right)\right) < -\log\left(1 - x_1 \exp\left(\sum_j \tilde{y}_j\right)\right)$$

Proof. Just notice that the Taylor coefficients of e^x and $-\log(1-x)$ are all positive.

Therefore,³

$$\sum_{I \in \mathcal{C}(\mathcal{P}): I \ni Q} w_I < -\log\left(1 - w_Q \exp\left(\sum_{I^*} \tilde{w}_{I^*}\right)\right)$$

and finally, using (12), monotonicity of $-\log(1-x)$, and the definition of L given in (4),

$$\sum_{I \in \mathcal{C}(\mathcal{P}): I \ni Q} |w_I| \leq -\log\left(1 - |w_Q| \exp\left(\sum_{I^*} |w_{I^*}|\right)\right) \leq L |w_Q| e^{a(Q)} \quad (13)$$

which proves the inductive step for the desired bound (7). We recall that (8) then follows from (7) by summing the bound (7) over all $\tilde{Q} \in \mathcal{P}$ incompatible with Q , because any index I incompatible with Q contains at least one polymer \tilde{Q} incompatible with Q .

2. PARTIALLY EXPANDED POLYMER MODELS. GENERALIZED D-K-P CRITERION

Consider now a more general “interacting” polymer model where in addition to polymers $P \in \mathcal{P}$ (the “big ones,” satisfying some compatibility relation) having weights w_P one also has a “cluster field,” i.e., a collection of complex fugacities $\{w_G, G \in \mathcal{G}\}$ indexed by objects G which we will call

³ Recall that the expression $\log(1 + w_Q \exp(-\sum_{I^*} w_{I^*}))$ which is a function of variables w_Q and $\{w_{I^*}\}$ is identified then also as a function of variables w_Q and $\{w_P; P \perp Q\}$.

“chains.” Assume that a relation $G \wr P$ resp. $G \subset P$ of (in)compatibility between the chains and polymers is given. On the contrary, all chains defined to be compatible with each other.

Given complex fugacities $\{w_P\}$ and $\{w_G\}$ we define the “mixed” (in the terminology of refs. 25 and 26 partition function

$$Z \equiv Z_{\mathcal{P}, \mathcal{G}, w} = \sum_{\{P_1, \dots, P_n\}_c: P_i \in \mathcal{P}} \prod_{i=1}^n w_{P_i} \exp \left(\sum_{G: G \subset P_i \forall i} w_G \right) \quad (14)$$

where the first sum is again over all families $\{P_1, \dots, P_n\}_c$ of pairwise compatible polymers in \mathcal{P} and the second sum $\sum_G w_G$ is over all chains G compatible with any polymer P_i .

Remark. Mixed partition functions arise from partially expanded polymer models. Assume that we have a polymer ensemble of the form $\mathcal{P} = \mathcal{P}_l \cup \mathcal{P}_s$ (in most applications, these are the “large” and “small” polymers separated according to some criterion). Then

$$Z_{\mathcal{P}} = \sum_{\substack{P_1, \dots, P_n: P_i \in \mathcal{P}_l, Q_1, \dots, Q_m: Q_i \in \mathcal{P}_s \\ \{P_1, \dots, P_n, Q_1, \dots, Q_m\}_c}} \prod_{i=1}^n w_{P_i} \prod_{j=1}^m w_{Q_j}$$

Here the entire collection $\{P_1, \dots, P_n, Q_1, \dots, Q_m\}$ must be compatible. For a given collection P_1, \dots, P_n , we can now take the logarithm of the sum

$$\log \left(\sum_{\substack{Q_1, \dots, Q_m: Q_i \in \mathcal{P}_s \\ \{P_1, \dots, P_n, Q_1, \dots, Q_m\}_c}} \prod_{j=1}^m w_{Q_j} \right) = \sum_{G: G \subset P_i \forall i} w_G$$

according to the procedure explained in the previous section, the notion of compatibility of the cluster G with a polymer P_i meaning that each element (small polymer) of the cluster is compatible with P_i . The result of this procedure is precisely a mixed partition function as defined above. In most applications one now wants to further expand another subclass of the “large polymers” that remain. To do this, one must be able to compute the logarithm of the mixed polymer partition function.

We now investigate the Taylor series of the logarithm of partition function (14)

$$\log Z = \sum_{I \in \mathcal{I}(\mathcal{P} \cup \mathcal{G})} w_I \quad \text{with} \quad w_I = C_I \prod_{P \in \text{supp } I} w_P^I \prod_{G \in \text{supp } I} w_G^I \quad (15)$$

where the sum is over the collection $\mathcal{I} = \mathcal{I}(\mathcal{P} \cup \mathcal{G})$ of all multi-indices I (integer valued functions on $\mathcal{P} \cup \mathcal{G}$) and C_I are given like in (3).

It can be shown, similarly as in the previous section that nonzero C_I appear only for “connected,” indecomposable multi-indices I called *clusters*. The *decomposability* of a multi-index with a support $\mathcal{P} \cup \mathcal{G} = (\mathcal{P}_1 \cup \mathcal{G}_1) \cup (\mathcal{P}_2 \cup \mathcal{G}_2)$ means that any polymer resp. chain from the first system is compatible with any one from the second one. Since all the chains G are mutually compatible, incompatibility can only occur between two polymers or between a polymer and a chain.⁴

Here, the *support* of a multi-index I is defined as

$$\text{supp } I = \{P: I_P > 0\} \cup \{G: I_G > 0\}$$

Denote by $\mathcal{C}(\mathcal{P} \cup \mathcal{G})$ the collection of all clusters on $\mathcal{P} \cup \mathcal{G}$.

We define the functions

$$L \equiv L(\delta) = \frac{-\log(1-\delta)}{\delta}, \quad E \equiv E(\delta) = \frac{e^\delta - 1}{\delta}, \quad \tilde{L} \equiv \tilde{L}(\delta) = L^2 E$$

Note that with these definitions we have, for all $x \geq 0$,

$$1 + LE(e^{Lx/\tilde{L}} - 1) \leq e^x \quad (16)$$

Theorem 2. Assume that there are functions $\{a(G), b(G) > 0, G \in \mathcal{P} \cup \mathcal{G}\}$ such that

$$|w_P| e^{a(P)} \leq \delta, \quad |w_G| \leq \delta, \quad \text{and} \quad (e^{|w_G|} - 1)(e^{b(G)} - 1) \leq \delta \quad (17)$$

holds for any polymer $P \in \mathcal{P}$ resp. chain $G \in \mathcal{G}$, and assume that the following two bounds hold:

(1) Polymer fugacities satisfy a bound, for any *chain or polymer* $G \in \mathcal{G} \cup \mathcal{P}$

$$\sum_{P \in \mathcal{P} \setminus G: P \uparrow G} |w_P| e^{a(P)} \leq \frac{b(G)}{\tilde{L}} \quad (18)$$

(2) Chain fugacities w_G satisfy a bound

$$\sum_{G \in \mathcal{G}: G \uparrow Q} |w_G| e^{b(G)} \leq a(Q) - b(Q) \quad (19)$$

for any *polymer* $Q \in \mathcal{P}$.⁵

⁴ A collection $\mathcal{P} \cup \mathcal{G}$ is a cluster if the graph whose bonds are pairs $P \uparrow P'$ and $P \uparrow G$ is *connected*.

⁵ Notice that while in (19) the sum is over chains incompatible with a *polymer* G , the sum in (18) is over polymers incompatible with a *polymer or chain* G .

Then the following bounds are valid:

(a) For the sum of all clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ containing a *polymer* $Q \in \mathcal{P}$

$$\sum_{I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G}): I \ni Q} |w_I| \leq L |w_Q| e^{a(Q)} \quad \text{where} \quad w_I = C_I \prod_{P \in \text{supp } I} w_P^{I_P} \quad (20)$$

(b) For the sum of all clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ containing a *chain* G

$$\sum_{I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G}): I \ni G} |w_I| \leq |w_G| e^{b(G)} \quad (21)$$

(c) For the sum of all clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ incompatible with a *chain* $G \in \mathcal{C}$

$$\sum_{I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G}): I \not\ni G} |w_I| \leq \frac{L}{\tilde{L}} b(G) \leq b(G) \quad (22)$$

(d) For the sum of all clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ incompatible with a *polymer* $Q \in \mathcal{P}$

$$\sum_{I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G}): I \not\ni Q} |w_I| \leq a(Q) \quad (23)$$

Proof. As before, we will use the induction over the total number $N = |\mathcal{P}| + |\mathcal{G}|$ of polymers and chains used in the model. The case $N = 0$ is trivial.

(a) Proof of (20): This is an analogy of (7). Denote by Z the partition function of the “full model,” by $Z_{\mathcal{P} \setminus Q, \mathcal{G}}$ the partition function of the model with polymer Q removed (i.e., with $w_Q = 0$). We have the relation (here and below, $Z \equiv Z_{\mathcal{P}, \mathcal{G}}$)

$$\begin{aligned} \sum_{I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G}): I \ni Q} w_I &= \log Z_{\mathcal{P}, \mathcal{G}} - \log Z_{\mathcal{P} \setminus Q, \mathcal{G}} \\ &= \log \left(1 + w_Q \exp \left(- \sum_{I^*} w_{I^*} \right) \right) \end{aligned} \quad (24)$$

where the sum $\sum_{I^*} w_{I^*}$ is precisely over clusters I^* from the $\mathcal{P} \setminus Q$ model incompatible with Q . We proceed analogously as in the proof (9)–(11) of (7), but using the bound (18) instead of (6). By the induction assumption

(23) for the $\mathcal{P} \setminus Q$ model, we already have $\sum_{I^*} |w_{I^*}| \leq a(Q)$. Then, by the definition of L , we get the required induction step

$$\sum_{I \in \mathcal{C}(\mathcal{P}): I \ni Q} |w_I| \leq -\log \left(1 - |w_Q| \exp \left(\sum_{I^*} |w_{I^*}| \right) \right) \leq L |w_Q| \exp(a(Q)) \quad (25)$$

(b) Denote by $Z_{\mathcal{P}, \mathcal{G} \setminus G}$ resp. $Z_{\mathcal{P} \setminus G, \mathcal{G}}$ the partition function of the model with G removed resp. partition function of the model where only polymers compatible with the chain G are allowed. For both models we may assume the validity of (20)–(23) by the induction assumption, namely we also have $Z_{\mathcal{P} \setminus G, \mathcal{G}} = e^{w_G} Z_{\mathcal{P} \setminus G, \mathcal{G} \setminus G}$. Notice that the term $\sum_{I \in \mathcal{P} \cup \mathcal{G}: I \ni G} w_I$ equals

$$\begin{aligned} \sum_{I: I \ni G} w_I &= \log Z - \log Z_{\mathcal{P}, \mathcal{G} \setminus G} \\ &= \log(Z_{\mathcal{P}, \mathcal{G} \setminus G} + (e^{w_G} - 1) Z_{\mathcal{P} \setminus G, \mathcal{G} \setminus G}) - \log Z_{\mathcal{P}, \mathcal{G} \setminus G} \\ &= \log \left(1 + (e^{w_G} - 1) \frac{Z_{\mathcal{P} \setminus G, \mathcal{G} \setminus G}}{Z_{\mathcal{P}, \mathcal{G} \setminus G}} \right) \\ &= \log \left(1 + (e^{w_G} - 1) \exp \left(- \sum_{I^*: I^* \ni P, P \perp G} w_{I^*} \right) \right) \end{aligned}$$

i.e.,

$$\sum_{I: I \ni G} w_I = w_G + \log \left(1 + (1 - e^{-w_G}) \left(\exp \left(- \sum_{I^*: I^* \ni P, P \perp G} w_{I^*} \right) - 1 \right) \right) \quad (26)$$

where the sum $\sum_{I^*} w_{I^*}$ is over clusters I^* containing a *polymer* P incompatible with G . By the induction hypothesis (22) we have $\tilde{L} \sum_{I^*} |w_{I^*}| \leq Lb(G)$ and we can continue in the estimate (of the sum of absolute values in the expansion of the r.h.s. of (26))

$$\begin{aligned} \sum_{I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G}): I \ni G} |w_I| &\leq |w_G| - \log \left(1 - (e^{|w_G|} - 1) \left(\exp \left(\sum_{I^*: I^* \ni P, P \perp G} |w_{I^*}| \right) - 1 \right) \right) \\ &\leq |w_G| - \log(1 - (e^{|w_G|} - 1)(e^{(L/\tilde{L})b(G)} - 1)) \\ &\leq |w_G| (1 + LE(e^{(L/\tilde{L})b(G)} - 1)) \leq |w_G| e^{b(G)} \end{aligned} \quad (27)$$

by (17) and (16). This proves (21).

(c) If a cluster I is incompatible with a chain G then there is some polymer $Q \in \mathcal{P}$ incompatible with G such that $I \ni Q$. Summing the r.h.s of (20) in (18), over all such $Q \ni G$ we arrive to (22) analogously as from (6) to (8).

(d) If a cluster I is incompatible with a polymer Q then (i) either I contains a polymer \tilde{Q} incompatible with Q (ii) or I contains a chain G incompatible with Q . The sum $\sum_I^{(i)} |w_I|$ corresponding to the first case is bounded as $\leq (L/\tilde{L}) b(Q)$ just by inserting (20) into (18). Analogously, the sum $\sum_I^{(ii)} |w_I|$ corresponding to the second case is bounded as $\leq a(Q) - b(Q)$, by inserting (21) into (19). Thus we get the desired bound (23).

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